Asymptotic High Energy Total Cross Sections and Theories with Extra Dimensions

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The rate at which cross sections grow with energy is sensitive to the presence of extra dimensions in a rather model-independent fashion. We examine how rates would be expected to grow if there are more spatial dimensions than 3 which appear at some energy scale, making connections with black hole physics and string theory. We also review what is known about the corresponding generalization of the Froissart-Martin bound and the experimental status of high energy hadronic cross sections which appear to saturate it up to the experimentally accessible limit of 100 TeV. We discuss how extra dimensions can be searched for in high energy cross section data and find no room for large extra dimensions in present data. Any apparent signatures of extra dimensions at the LHC would seem to have to be interpreted as due to some other form of new physics.

I. INTRODUCTION

There is a wealth of possible constraints on new physics which can be obtained without the need to look for specific exclusive processes which may be difficult to detect and distinguish from backgrounds simply by looking at how cross sections grow with energy. We review simple arguments for how cross sections are expected to grow with energy, and connect them with black hole thermodynamics and string-theoretical models. Allowing for a number of space dimensions greater than 3 we find, in agreement with earlier discussions of the Froissart-Martin bound in higher dimensions, much faster growth of cross sections with energy than is allowed by unitarity in 3+1 dimensions. Since the experimental results saturate the unitarity bounds, we find there is no room for extra dimensions[1] at scales below 100 TeV and point out that the best way to search for such dimensions may well be via measurements of the energy dependence of cross sections rather than via searches for exclusive processes.

II. HIGH ENERGY TOTAL CROSS SECTION

A clear physical argument[2] for predicting the high center of mass energy E total cross section is the following: (i) If the elastic scattering amplitude at high energy is dominated by the exchange of the lightest mass μ particle, then the probability of the exchange in space reads

$$P(r, E) \sim \exp\left[-\frac{2\mu c r}{\hbar} + \frac{S(E)}{k_B}\right], \qquad (1)$$

where $2\mu r/\hbar$ is the WKB factor for a particle to move a space-like distance r and the *entropy* $S(E)$ determines the density of final states. (ii) The probability becomes of order unity at a distance

$$R = \frac{\hbar S(E)}{2\mu c k_B} . \qquad (2)$$

(iii) The total cross section for *shadow scattering* is then given by $\sigma_{\text{total}} = 2\pi R^2$ yielding

$$\sigma_{\text{total}}(E) = \frac{\pi}{2}\left[\frac{\hbar}{\mu c}\right]^2\left[\frac{S(E)}{k_B}\right]^2. \qquad (3)$$

The asymptotic $E \to \infty$ total cross section is thereby determined by the entropy $S(E)$.

A typical entropy estimate may be made via the following reasoning: The equipartition theorem for a gas of ultra-relativistic particles implies a mean particle energy varying linearly with temperature; $\overline{\epsilon} \approx \overline{c|\mathbf{p}|} \approx 3k_B T$. A Boltzmann gas of such particles has a constant heat capacity. A system with a constant heat capacity C_∞ obeys

$$E = C_\infty T = C_\infty \frac{dE}{dS} \quad \Rightarrow \quad dS = C_\infty \frac{dE}{E} , \qquad (4)$$

yielding an entropy logarithm

$$S(E) = C_\infty \ln\left(\frac{E}{E_0}\right). \qquad (5)$$

By virtue of Eqs.(3) and (5), the total cross section for a constant heat capacity system,

$$\sigma_{\text{total}}(E) = \frac{\pi}{2}\left[\frac{\hbar C_\infty}{\mu c k_B}\right]^2 \ln^2\left(\frac{E}{E_0}\right), \qquad (6)$$

saturates the Froissart-Martin bound[3, 4]. In a more general thermodynamically stable situation, the entropy $S(E)$ is determined parametrically by the heat capacity

as a function of temperature

$$C(T) = \frac{dE(T)}{dT} = T\frac{dS(T)}{dT} \ ,$$

$$E = \int_0^T C(T')dT',$$

$$S = \int_0^T C(T')\frac{dT'}{T'} \ . \tag{7}$$

The saturation Eqs.(5) and (6) will then hold true only in the high anergy and high temperature limit of a stable heat capacity $C(T \to \infty) = C_\infty$.

To compute the total high energy cross section for models with extra dimensions, the central theoretical problem is to understand the entropy implicit in such models. In so far as extra dimensional theories are thought to describe the gravitational interaction, the resulting entropy may be expected to exhibit a *second law violation*. Let us first review this second law instability for the well known case of the black hole entropy.

III.   BLACK HOLE ENTROPY

The gravitational coupling strength of a black hole with a mass M may be defined via

$$\alpha_G(M^2) = \frac{GM^2}{\hbar c} = \frac{S}{4\pi k_B} \ , \tag{8}$$

wherein S is the black hole entropy, $E = Mc^2$ is the black hole energy and

$$S = 4\pi k_B \left(\frac{G}{\hbar c^5}\right) E^2. \tag{9}$$

The black hole temperature is then

$$\frac{1}{T} = \frac{dS}{dE} \quad \Rightarrow \quad E = 2\pi\tau_G \left(\frac{\hbar c}{k_B T}\right), \tag{10}$$

where the gravitational vacuum tension τ_G is determined by

$$2\pi\tau_G = \frac{c^4}{8\pi G} \approx 4.816 \times 10^{42} \ \text{Newton}. \tag{11}$$

The physical meaning of the vacuum gravitational tension becomes evident if the Einstein field equations for the energy-pressure tensor are written in the form

$$T_{\mu\nu} = 2\pi\tau_G \left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]. \tag{12}$$

The Planck mass M_P is defined via Eq.(8) employing $\alpha_G(M_P^2) = 1$,

$$M_P = \sqrt{\frac{\hbar c}{G}} \approx 6.88 \times 10^{-5}\text{gm} \approx 3.86 \times 10^{28} eV/c^2. \tag{13}$$

Note that the energy,

$$\frac{E}{M_P c^2} = \frac{M_P c^2}{8\pi k_B T}, \tag{14}$$

decreases as the temperature increases in violation of the second law of thermodynamics, i.e. the black hole heat capacity $C = dE/dT < 0$. This instability is a common feature of classical self-gravitationally bound systems.

IV.   HAGEDORN-STRING ENTROPY

Theories with extra dimensions grew out of string theories of interactions which would include in principle gravitational theories. A classical picture of a rotating string can be developed. If one considers a string of length L, then the lowest resonance frequency obeys

$$\omega = \frac{\pi c}{L} \ . \tag{15}$$

Treating the resonant frequency ω as an angular velocity conjugate to the string angular momentum J one finds

$$\omega = \frac{dE}{dJ} \tag{16}$$

where the energy is related to the string tension τ_S via

$$E = \tau_S L. \tag{17}$$

Eqs.(15), (16) and (17) imply the differential equation,

$$dJ = \frac{EdE}{\pi c\tau_S} \ , \tag{18}$$

whose solution is the linear Regge trajectory,

$$J = \hbar \left(\alpha_0 + \alpha' E^2\right). \tag{19}$$

In Eq.(19), α_0 is the Pomeranchuck intercept[5, 6] and

$$2\pi\tau_S = \frac{1}{\hbar c\alpha'} \tag{20}$$

is the generalized string tension analog of the gravitational tension Eq.(11) which is now determined by the Regge trajectory slope parameter α'. Let us consider the entropy consequences of these types of models.

A.   Hagedorn Temperature

The canonical partition function for a system in an environmental temperature $\tilde{T}$ has the form

$$Z(\tilde{T}) = \sum_E \Omega(E) \exp(-E/k_B\tilde{T}), \tag{21}$$

where $\Omega(E)$ is the number of quantum states with energy E. In terms of the entropy

$$S(E) = k_B \ln \Omega(E) \tag{22}$$

and the canonical free energy

$$F(\tilde{T}) = -k_B \tilde{T} \ln Z(\tilde{T}), \qquad (23)$$

one finds from Eqs.(21), (22) and (23) that

$$\exp\left[-\frac{F(\tilde{T})}{k_B \tilde{T}}\right] = \sum_E \exp\left[\frac{S(E)}{k_B} - \frac{E}{k_B \tilde{T}}.\right]. \qquad (24)$$

The Hagedorn string entropy has the form[7, 8]

$$S(E) = \frac{E}{T_H} - k_B \eta \ln\left(\frac{E}{k_B T_H}\right) + k_B \gamma, \qquad (25)$$

wherein η and γ are dimensionless constants and T_H is the Hagedorn temperature.

B. Statistical Thermodynamics

In order that the partition function in Eq. (21) converge, the environmental temperature $\tilde{T}$ must be less than the Hagedorn temperature T_H.

$$\tilde{T} < T_H , \qquad (26)$$

On the other hand, the string temperature T given by,

$$\frac{1}{T} = \frac{dS}{dE}, \qquad (27)$$

yields a positive energy

$$E = \eta k_B T_H \left[\frac{T}{T - T_H}\right] \equiv \Phi\left[\frac{T}{T - T_H}\right] > 0, \qquad (28)$$

Eq.(28) implies a string temperature larger than the Hagedorn temperature,

$$T > T_H , \qquad (29)$$

as well as a threshold for the string excitation energy

$$E > \Phi = \eta k_B T_H, \qquad (30)$$

Note the inequalities

$$\tilde{T} < T_H < T, \qquad (31)$$

which imply an environmental temperature $\tilde{T}$ which is less tha the string temperature T. That a thermodynamically stable environment cannot come into equilibrium with a string is due to the latter having negative heat capacity

$$C = \frac{dE}{dT} = -\Phi\left[\frac{T_H}{(T - T_H)^2}\right] < 0 \qquad (32)$$

which is a thermodynamic second law violation.

C. String Theory

The extra dimension value n for space-time is related to the full dimension of space-time $\mathcal{D}$ via

$$\mathcal{D} = 3 + 1 + n = 4 + n. \qquad (33)$$

The Hagedorn entropy for string theories is illustrated by bosonic string example where the entropy can be computed from the number theory integer partition functions[9]. It is

$$\frac{S(E)}{k_B} = 2\pi E\sqrt{\alpha'\left[\frac{\mathcal{D} - 2}{6}\right]}$$
$$- \left(\frac{\mathcal{D} + 1}{2}\right)\ln(\sqrt{\alpha'}E)$$
$$\left(\frac{\mathcal{D} - 1}{4}\right)\ln\left(\frac{\mathcal{D} - 2}{24}\right) - \frac{\ln 2}{2}, \qquad (34)$$

which is clearly a special case of the general Hagedorn string entropy Eq.(25). For the supersymmetric string which includes fermions, the functional form of the Hagedorn entropy in Eq.(25) is expected to remain intact. However, the detailed dependence of the parameters T_H, η and γ on $\mathcal{D}$ is expected to depend on the specific string model chosen.

V. COMPACT EXTRA DIMENSIONS

Let us suppose that n extra dimensions are compact with a gravitational length scale Λ. For distance r large on the length scale Λ, the Newtonian attraction is determined by the gravitational string tension in Eq.(11);

$$-U = G\frac{m_1 m_2}{r}$$
$$= \frac{1}{16\pi^2}\left[\frac{(m_1 c^2)(m_2 c^2)}{\tau_G r}\right]$$
$$= \frac{1}{16\pi^2}\left[\frac{(m_1 c^2)(m_2 c^2)}{\sigma_n \Lambda^n r}\right] \quad (r \gg \Lambda). \qquad (35)$$

The nature of adding n compact extra dimensions is illustrated schematically in FIG.1. For short distances, the power law energy has an exponent

$$-U \propto \left[\frac{(m_1 c^2)(m_2 c^2)}{\sigma_n r^{1+n}}\right] \quad (r \ll \Lambda). \qquad (36)$$

Similar considerations apply for the string entropy with n extra dimansions each of length Λ. For a gravitational string one may begin with an entropy of the Hagedorn form as given in Eq.(25); i.e. as $E \to \infty$

$$\frac{S_0(E)}{k_B} = \varpi_0\left[\frac{E}{\sqrt{\hbar c \tau_G}}\right] + \cdots, \qquad (37)$$

$$\tau_G r = \sigma_0 r$$

$$\tau_G r = \sigma_1 \Lambda r$$

Λ

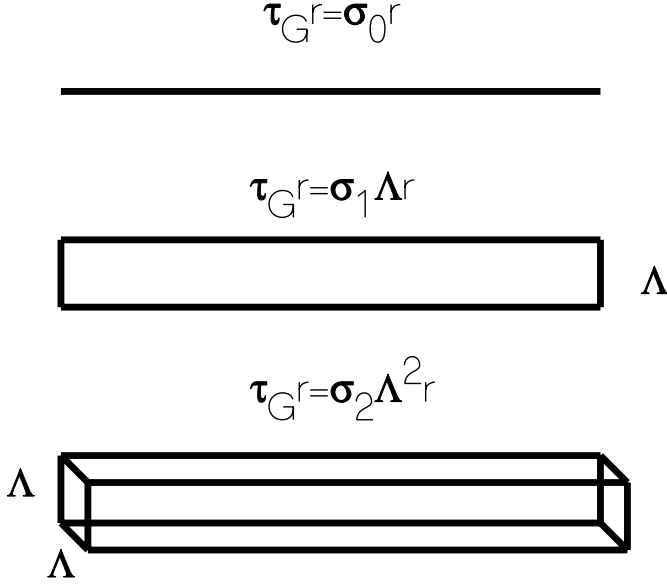$$\tau_G r = \sigma_2 \Lambda^2 r$$

Λ

Λ

FIG. 1: For a one dimenisional string of length r under gravitational tension, $\tau_G \equiv \sigma_0$, the energy of stretching $\tau_G r$ enters into Newton's gravitational law as in the denominator of Eq.(35). If the string has one extra dimension of length $\Lambda \ll r$, then the stretching two dimensional membrane is described by a surface tension $\sigma_1 = \tau_G/\Lambda$. If the string has two extra dimensions each of length $\Lambda \ll r$, then the stretching three dimensional square prism is described by a negative pressure $\sigma_2 = \tau_G/\Lambda^2$. For the case of $n \geq 1$ added dimensions, one finds $\sigma_n = \tau_G/\Lambda^n$ as in Eq.(35).

wherein ϖ_0 is a dimensionless constant of order unity. With n extra compact dimensions

$$\frac{S_n(E)}{k_B} = \varpi_n \left[\frac{E}{\sqrt{\hbar c \sigma_n}} \right] \left(\frac{E}{\hbar c} \right)^{n/2} + \cdots$$
$$= \varpi_n \sqrt{\frac{\hbar c}{\sigma_n}} \left(\frac{E}{\hbar c} \right)^{1+(n/2)} + \cdots . \qquad (38)$$

This can be compared with the string scattering results of Amati, Ciafaloni and Veneziano who find for the total, diffractive and inclusive cross sections σ_{tot}, σ_{dif} and σ_{inel} respectively

$$\sigma_{tot} \sim s^{\frac{D-2}{D-4}}$$
$$\sigma_{dif} \sim s$$
$$\sigma_{inel} \sim (\ln(s))^{D-2}$$

at high energies, where D is the number of dimensions of spacetime.

We now discuss how extra compact dimensions have an effect on the total hadron-hadron cross section in the limit of high center of mass energy.

VI. TOTAL HADRONIC CROSS SECTION

From the relationship between the total cross section entropy in Eqs.(3), (5) and (38), one finds the total cross

section in the high energy limit

$$\sigma_{\text{total}}(s) = \frac{\pi}{2} \left[\frac{\hbar}{\mu c} \right]^2 \times$$

$$\left| \frac{C}{k_B} \ln \left(\frac{s}{s_0} \right) + \varpi_n \sqrt{\frac{\hbar c}{\sigma_n}} \, s^{(n+2)/4} + \cdots \right|^2 \qquad (39)$$

wherein the center of mass energy $E = \hbar c \sqrt{s} \to \infty$. Below the threshold for the observation of extra dimensions

$$\left[s_0 \ll s \ll \left(\frac{\sigma_n}{\hbar c} \right)^{2/(2+n)} \right],$$

$$\sigma_{\text{total}}(s) = \frac{\pi}{2} \left[\frac{C\hbar}{k_B \mu c} \right]^2 \ln^2 \left(\frac{s}{s_0} \right) . \qquad (40)$$

Above the threshold for the observation of extra dimensions,

$$\left[\left(\frac{\sigma_n}{\hbar c} \right)^{2/(2+n)} \ll s \right],$$

$$\sigma_{\text{total}}(s) = \frac{\pi \varpi_n^2 \hbar c}{2\sigma_n} \left[\frac{\hbar}{\mu c} \right]^2 s^{(n+2)/2}. \qquad (41)$$

The transition from Eq.(40) to (41) at the energy

$$E_{\text{transition}} \sim \hbar c \left(\frac{\sigma_n}{\hbar c} \right)^{1/(2+n)}$$

$$\sim \left(\frac{\tau_G}{\hbar c} \right)^{1/(2+n)} \frac{\hbar c}{[\Lambda^n]^{1/(n+2)}} \qquad (42)$$

which depends on both the number n of extra dimensions and the compact length scale Λ.

VII. THE FROISSART BOUND IN HIGHER DIMENSIONS

The discussions above have been quite physical, but one can also directly generalize the Froissart bound by repeating the original argument but now in more than 3 spatial dimensions. This has been done by Chaichian and Fisher[10] who find that cross sections must be bounded by higher powers of $\ln(s)$ or even as powers of s multiplied by logarithms.

VIII. GENERIC EFFECTS OF HIGHER DIMENSIONS

A key observation of this paper is that if one assumes that there are extra dimensions of space, one generically expects a qualitative change in how cross sections rise with energy. That change is that they rise significantly faster than one would expect in 3 spatial dimensions as soon energies rise high enough that those extra dimensions are accessible. This is an essentially model-independent statement, and due physically to the fact

the adding dimensions opens up phase space for decays. This is true whether or not the Standard Model matter particles are restricted to lie on some 3-dimensional "brane"[1]. As long as one can excite degrees of freedom which see more than 3 space dimensions, the rate at which cross sections grow will be higher than one would expect in 3. Note that such behavior could even be expected simply on dimensional grounds, with the material presented above being a clarification of this point.

Now it is well-known that hadronic cross sections saturate the unitarity bounds so one could search for evidence of extra dimensions simply by search for energies above which cross sections grow faster than with energy than at lower dimensions. A detailed fit with two functional forms above and below $E_{transition}$ as defined above could be used to clarify the nature and number of such extra dimensions, should any evidence for them at all arise. However, as we show in the following section, extra dimensions seem to be ruled out up to scales of about 100 TeV already from the present data.

IX. CONSTRAINTS ON COMPACT DIMENSIONS FROM HIGH ENERGY CROSS-SECTION DATA

The experimental situation on high-energy hadronic cross-sections may be summarized as follows. High quality data exist up to $\sqrt{s} = 2$ TeV for $\sigma_{total}^{(p\bar{p})}(s)$ from the Tevatron and up to $\sqrt{s} = 0.2$ TeV for $\sigma_{total}^{(pp)}(s)$. Less precise data from cosmic rays are available for $\sigma_{total}^{(pp)}(s)$ up to $\sqrt{s} = 100$ TeV. Recently, the ATLAS group at LHC has released preliminary data on $\sigma_{inelastic}^{(pp)}(s)$ at $\sqrt{s} = 7$ TeV.

There are three reasonably complete phenomenological analyses of the above cross-section data which may be summarized as follows:

- PDG[12]: They *assume* that $\sigma_{total}(s)$ rises as $(ln(s/s_o))^2$, the *maximum* rise allowed by the Froissart bound. Their parametrization for all the available data reads

$$\sigma_{total}^{(pp,p\bar{p})}(s) = (0.308\text{mb}) \, [\ln(\frac{s}{\text{GeV}^2})]^2 \\ +(35.45\text{mb}) + ..., \qquad (43)$$

the dots meaning non-leading terms vanishing at high energy. A $\chi^2/\text{d.o.f.} = 1.05$ -for all data considered- is given.

- BH[13]: Block and Halzen also find strong evidence that the Froissart bound is saturated. That is, their result for a generic hadronic cross-section (in the

asymptotic domain) may be written as

$$\sigma_{total}^{(pp,p\bar{p})}(s) = C_1[\ln(\frac{s}{\text{GeV}^2})]^2 \\ + C_2[\ln(\frac{s}{\text{GeV}^2})] \\ +C_3 + ..., \qquad (44)$$

where the constants $C_i, \ i = 1, 2, 3$ are determined from the data.

- GGPS[14]: This is an independent theoretical analysis -developed over two decades- which is based on mini-jets, incorporating aspects of confinement and soft-gluon resummation. It has been quite successful in obtaining a rather complete description of total cross-sections for $pp, p\bar{p}, \pi p, \gamma p, \gamma\gamma$ processes. This group finds the rise to be $(\ln(s/s_o))^q$ where $1 < q < 2$, with a phenomenological value: $4/3 < q < 3/2$.

Thus, *all* viable phenomenological analyses of total and inelastic cross-sections up to $\sqrt{s} = 100$ TeV conclude that

$$\sigma(s) \rightarrow \sigma_o[ln(s/s_o)]^q, \quad with \ q \leq 2. \qquad (45)$$

Eq.(45) is at complete odds with Eq.(41) even for $n = 1$. Hence, there appears to be no room for any compactified dimensions whose thresholds are below 100 TeV.

X. CONCLUSIONS

We argue on general grounds, and via concrete calculations, that if extra space dimensions are accessible to at least some excitations produced in high energy hadronic collisions above some energy scale $E_{critical}$ one should see an increase in the rate at which total cross sections rise. This suggests a general means to search for such extra dimensions by looking at cross sections as a function of energy. Such searches are particularly attractive since they are highly model-independent and largely unaffected by single events which might, due to statistical fluctuations, seem to support some model with higher dimensions. Since the data so far apparently saturate the Froissart-Martin bound in 3+1 dimensions, there appears to be no room for any compactified dimensions whose thresholds are below 100 TeV. This limit can be extended as data from ever higher energy collisions are obtained, but for the near future this will have to come from cosmic ray data. Any events which might seem to be signals of extra spacetime dimensions at the LHC would have to be attributed to some other sort of new physics.

Since this paper was written, Block and Halzen [15](who have accepted our work with an aim to extending it) have suggested that higher dimensions might be ruled out to arbitrarily high energies via the same argument, but they use a Froissart bound which is tied to

3+1 dimensions, which as discussed above, is dimension-dependent. Further recent discussions include that of Fagundes, Menon and Silva [16, 17] (see also the commentary by Block and Halzen[18]) who suggest that energy dependence of hadronic total cross section at high energies may be an open problem still). Block and Halzen have also recently argued that the Auger cross section data supports the idea that proton asymptotically develops into a disk[19].

None of these publications changes the conclusions of our paper.

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